Lab 03-Fourier Series

In this lab session, we will introduce a way of analyzing/decomposing a continuous time signal into frequency components given by sinusoidal signals. This process in crucial in the signal processing field since it reveals the frequency content of signal and simplifies the calculation of systems' output. The analysis is based on the Fourier series. Up to this point, all signals were expressed in the time domain. With the use of Fourier series, a signal is expressed in the frequency domain and sometimes a frequency representation of a signal reveals more information about the signal than its time domain representation. There are three different and equal ways that can be used in order to express a signal into sum of simple oscillating functions, i.e., into a sum of sines, cosines, or complex exponentials. In this manual, symbols n and k are often swapped in order for the code written in examples to be in accordance with the theoretical mathematical equations.

3.1 Complex Exponential Fourier Series:

Suppose that a signal $x(t)$ is defined in the time interval $\left[t_0, t_0+T\right]$. Then, $x(t)$ is expressed in exponential Fourier series form (equation 3.1) as

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\Omega_0 t}, \qquad t\in\left[t_0, t_0+T\right]$$

where,

Ω_0 is the fundamental frequency, and is given by $\Omega_0=\left(2\pi/T\right)$

t_0, T are real numbers

The terms a_k that appear in equation 3.1 are given by equation 3.2 as

$$a_k=\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\Omega_0 t}dt$$

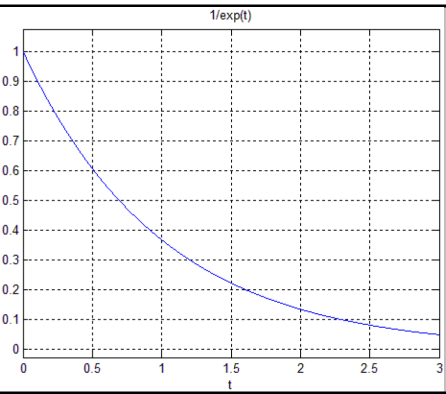
The complex coefficients a_k are called complex exponential Fourier series coefficients, while a_0 is a real number and is called a constant or dc component. Each coefficient a_k corresponds to the projection of the signal $x(t)$ at the frequency $k\Omega_0$, which is known as k^{th} harmonic. The Fourier series expansion is valid only in the interval $\left[t_0, t_0+T\right]$, and the value of T defines the fundamental frequency Ω_0. As the Fourier series coefficients represent the signal in the frequency domain, they are also referred as the spectral coefficients of the signal.

Example:

Expand in complex exponential Fourier series the signal $x(t)=e^{-t}, 0\le t\le 3$.

The first thing that need to be done is to define the quantities $t_0=0, T=3$, and $\Omega_0=\left(2\pi/T\right)$. Moreover, the signal $x(t)$ is defined as symbolic expression.

Commands	*Results*	*Comments*

`t0=0;`
`T=3;`
`w=2*pi/T;`
`syms t`
`x=exp(-t);`
`ezplot(x,[t0 t0+T]), grid on`
		Definition and graph of signal $x(t)$ in the time interval $[t_0, t_0+T]$

Afterwards, the coefficients a_k are computed according to equation 3.2. Looking into equation 3.1, we observe that Fourier coefficients a_k have to be calculated. Of course, this computation cannot be done in an analytical way. Fortunately, as the index k approaches toward $+\infty$ or toward $-\infty$, the Fourier coefficients a_k are approaching zero. Thus, $x(t)$ can be satisfactorily approximated by using a finite number of complex exponential Fourier series terms. Consequently, by computing the coefficients a_k for $-100 \le k \le 100$, i.e., by using first 201 complex exponential terms, a good approximation of $x(t)$ is expected. The approximate signal is denoted by $xx(t)$, and is computed by equation 3.3 given as

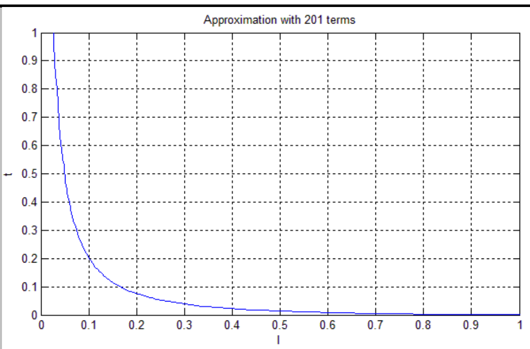
$$xx(t) = \sum_{k=-K}^{+K} a_k e^{jk\Omega_0 t}, \qquad t \in [t_0, t_0+T]$$

Commands	*Comments*

`for k=-100:100`
`    a(k+101)=(1/T)*int(x*exp(-*k*w*t),t,t0,t0+T)`
`end`
	Calculation of coefficients a_k according to equation 3.2.

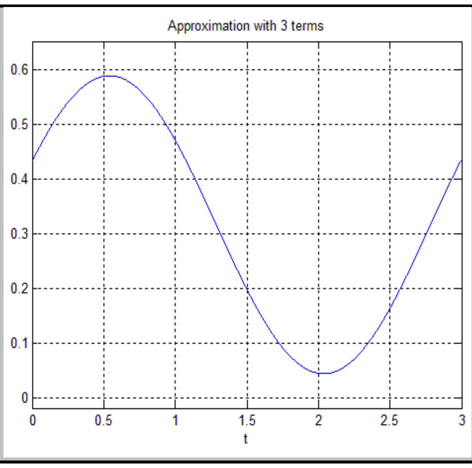
In order to define the vector a that contains the Fourier series coefficients a_k, $-100 \le k \le 100$, the syntax $a(k+101)$ is used for programming reasons, since in MATLAB the index of a vector cannot be zero or negative. Having calculated the coefficients a_k, the signal $x(t)$ is approximated according to equation 3.2, or more precisely according to equation 3.3. Note that equation 3.2 is sometimes called the *synthesis* equation, while we in this manual equation 3.3 is referred as *analysis* equation.

Commands	*Results*	*Comments*

`for k=-100:100`
`    ex(k+101)=exp(j*k*w*t);`
`end`
`xx=sum(a.*ex)`
`ezplot(xx,[t0 t0+T]), grid on`
`title('Approximation with 201 terms')`
		Initially the quantities $e^{jk\Omega_0 t}, -100 \le k \le 100$ are computed and afterward the signal is approximated according to equation 3.3.

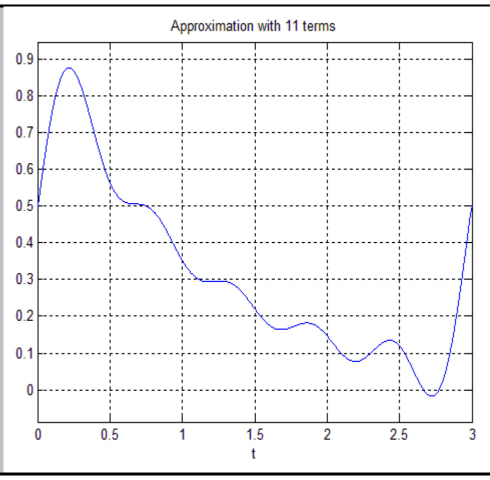
The plotted signal $xx(t)$ that is computed with the use of the complex exponential Fourier series is almost identical with the original signal $x(t)$. In order to understand the importance of number of terms used for approximation of original signal $x(t)$, the approximate signal $xx(t)$ is constructed for different values of k. First, the signal $x(t)$ is approximated by three exponential terms, i.e., the coefficients a_k are computed for $-1 \le k \le 1$.

Commands	*Results*	*Comments*

`clear a ex;`
`for k=-1:1`
 	
`a(k+2)=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);`
`end`
`for k=-1:1`
`    ex(k+2)=exp(j*k*w*t);`
`end`
`xx=sum(a.*ex);`
`figure();`
`ezplot(xx,[t0 t0+T]),`
`grid on`
`title('Approximation with 3 terms')`
		When 3 terms are used in approximation of $x(t)$ by $xx(t)$,i.e., $-1 \le k \le 1$, the approximation signal $xx(t)$ is pretty dissimilar from the original signal $x(t)$.

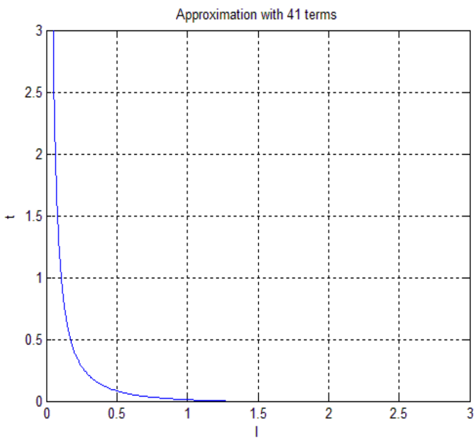
Next, the signal $x(t)$ is approximated by 11 exponential terms, i.e., the coefficients a_k are computed for $-5 \le k \le 5$.

Commands	*Results*	*Comments*

`for k=-5:5`
 	
`a(k+6)=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);`
`end`
`for k=-5:5`
`    ex(k+6)=exp(j*k*w*t);`
`end`
`xx=sum(a.*ex);`
`figure();`
`ezplot(xx,[t0 t0+T]),`
`grid on`
`title('Approximation with 11 terms')`
		It is clear that even when 11 terms are used, namely, $-5 \le k \le 5$, the approximation of $x(t)$ by $xx(t)$ is not good and is pretty dissimilar to $x(t)$.

Finally, the signal $x(t)$ is approximated by 41 exponential terms, i.e., the coefficients a_k are computed for $-20 \le k \le 20$.

Commands	*Results*	*Comments*

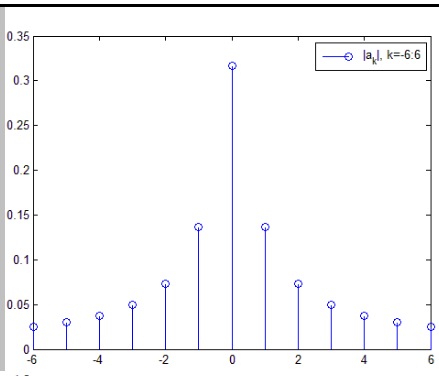
`for k=-20:20`
 	
`a(k+21)=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);`
`end`
`for k=-20:20`
 	
`ex(k+21)=exp(j*k*w*t);`
`end`
`xx=sum(a.*ex);`
`figure();`
`ezplot(xx,[t0 t0+T]),`
`grid on`
`title('Approximation with 41 terms')`
		The signal $xx(t)$ is now computed from 41 terms and is starting to look quite similar to the original signal $x(t)$. Thus, this is a quite satisfactory approximation.

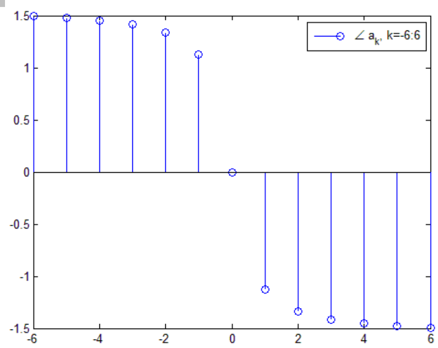
From the above analysis, it is clear that when many exponential terms are being considered in the construction of the approximate signal, a better approximation if the original signal is obtained. As illustrated in the beginning of the example, when 201 terms were used for the construction of the approximate signal, the obtained approximation was very good.

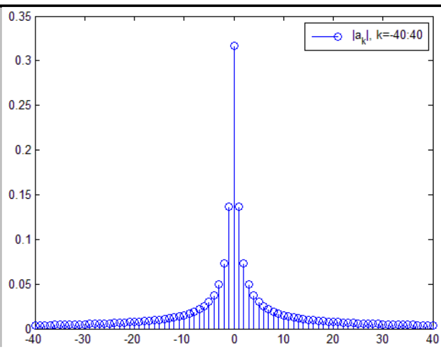
3.2 *Plotting Fourier Series coefficients:*

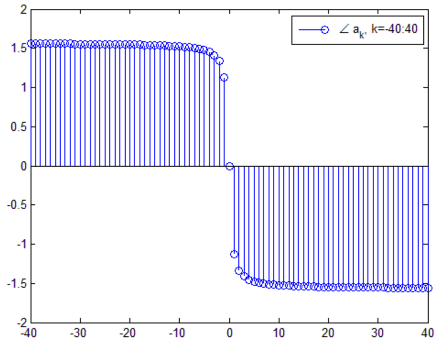
In the previous section, the Fourier coefficients were computed. In this section, the way of plotting them is presented. Once again, the signal the signal $x(t)=e^{-t}, 0 \le t \le 3$ is considered. The coefficients a_k of the complex exponential form are the first that will be plotted for $-6 \le k \le 6$ and for $-40 \le k \le 40$. In the usual case, the coefficients a_k are complex numbers. A complex number z can be expressed as $z=|z|e^{j\angle z}$, where $|z|$ is the magnitude and $\angle z$ is the angle of z. Therefore, in order to create the graph of the coefficients a_k of the complex exponential form, the magnitude and the angle of each coefficient have to be plotted.

Commands	*Results*	*Comments*

`syms t k n`
`x=exp(-t);`
`t0=0;`
`T=3;`
`w=2*pi/T;`
`a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);`
`k1=-6:6;`
`ak=subs(a,k,k1);`
`stem(k1,abs(ak));`
`legend('|a_k|, k=-6:6')`
		The signal $x=e^{-t}, 0 \le t \le 3$ is defined and the magnitude of the coefficients is plotted for $-6 \le k \le 6$.

`stem(k1,angle(ak));` `legend('\angle a_k, k=-6:6')`		The angles of coefficients are plotted for $-6 \le k \le 6$. It is worth noticing the way that the angle symbol is drawn through the `legend` command.

`k2=-40:40;`
`ak2=subs(a,k,k2);`
`stem(k2,abs(ak2));`
`legend('|a_k|, k=-40:40')`
	

`stem(k2,angle(ak2));` `legend('\angle a_k, k=-40:40')`	

3.3 *Fourier Series of Complex Signals:*

So far, we were dealing with real signals. In this section, we examine the Fourier series representation of a complex valued signal.

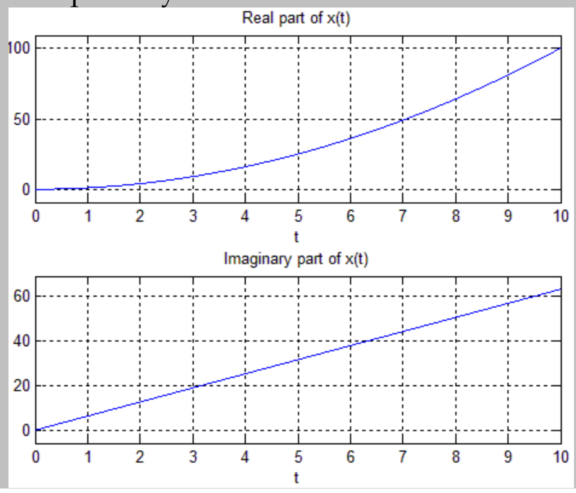
Example:

Compute the coefficients of the complex exponential Fourier series and the trigonometric Fourier series of the complex signal $x(t) = t^2 + j2\pi t, 0 \le t \le 10$. Moreover, plot the approximate signals using5 and 41 components of the complex exponential form.

Commands	*Results/Commands*

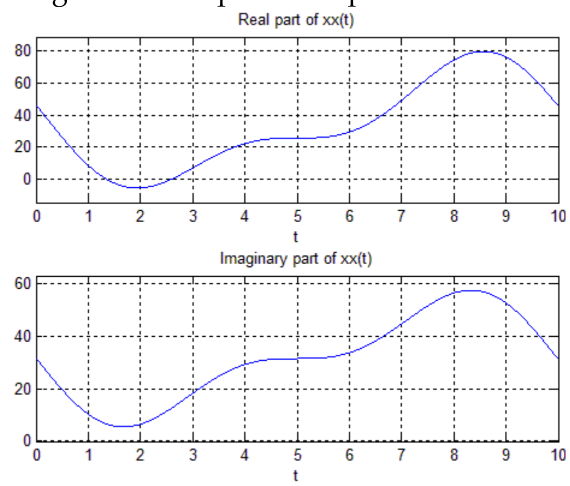
```
syms t
t0=0;
T=10;
w=2*pi/T;
x=t^2+j*2*pi*t;
subplot(211)
ezplot(real(x),[t0 T]),grid on;
title('Real part of x(t)');
subplot(212)
ezplot(imag(x),[t0 T]),grid on;
title('Imaginary part of x(t)');
```

The signal $x(t) = t^2 + j2\pi t$ is complex; hence, its real and imaginary parts are plotted separately.

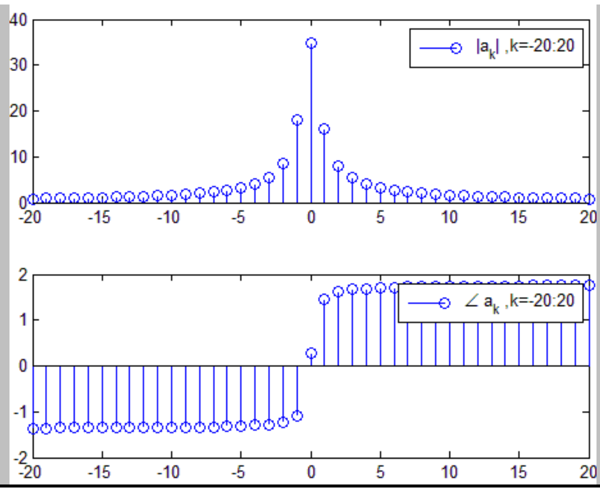



```
for k=-2:2
    a(k+3)=(1/T)*int(x*exp(-
j*k*w*t),t,t0,t0+T);
    ex(k+3)=exp(j*k*w*t);
end
xx=sum(a.*ex);
subplot(211)
ezplot(real(xx),[t0 T]),grid on;
title('Real part of xx(t)');
subplot(212)
ezplot(imag(xx),[t0 T]),grid on;
title('Imaginary part of xx(t)');
```

Computation of the first five coefficients of complex exponential form and graph of the approximate signal is obtained with five terms. The coefficients of the complex exponential form are complex, thus their magnitude and phase are plotted.




```
a1=eval(a)
subplot(211);
stem(-20:20,abs(a1));
legend ('|a_k| ,k=-20:20')
subplot(212);
stem(-20:20,angle(a1));
legend ('\angle a_k ,k=-20:20')
```



Evaluate the approximation by 41 components yourself.

3.3 *Fourier Series of Periodic Signals:*

In the previous sections, the Fourier series expansion of a signal was defined in a close time interval $[t_0, t_0 + T]$. Beyond this interval, the Fourier series expansion does not always converge to the original signal $x(t)$. In this section, we introduce the case where the signal $x(t)$ is a periodic signal with period T, i.e., $x(t) = x(t+T)$. In this case, the Fourier series is also periodic with period T; thus converges to $x(t)$ for $-\infty \leq t \leq +\infty$.

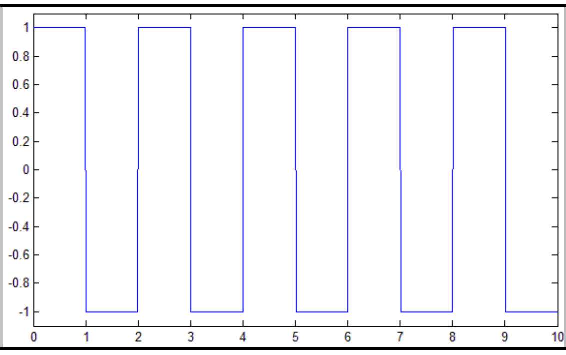
Example:

Approximate by Fourier series, the periodic signal that in one period is given by

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -1, & 1 \leq t \leq 2 \end{cases}$$

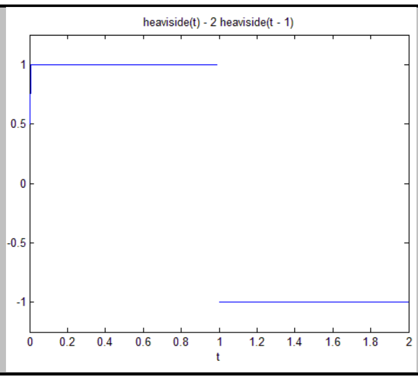
First the signal $x(t)$ is plotted over the time of five periods for reference reasons.

Commands	*Results/Comments*

`t1=0:.01:1;`
`t2=1.01:.01:2;`
`x1=ones(size(t1));`
`x2=-ones(size(t2));`
`x=[x1 x2];`
`xp=repmat(x,1,5);`
`t=linspace(0,10,length(xp));`
`plot(t,xp)`
	

Afterwards, the two part signal $x(t)$ is defined as single symbolic expression $x(t) = u(t) - 2u(t-1), 0 \leq t \leq 2$, where $u(t)$ is the unit step function. Note that the periodic signal $x(t)$ is entirely determined by its values over one period. Thus, the symbolic expression of $x(t)$ is only defined for time interval of interest, namely, $0 \leq t \leq 2$ (one period). The defined symbolic expression is plotted in one period for confirmation.

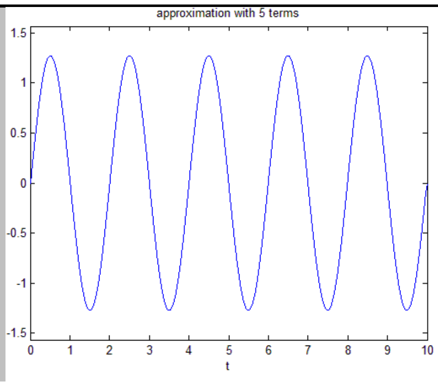
Commands	*Results*

`syms t`
`x=heaviside(t)-2*heaviside(t-1);`
`ezplot(x,[0 2]);`
	

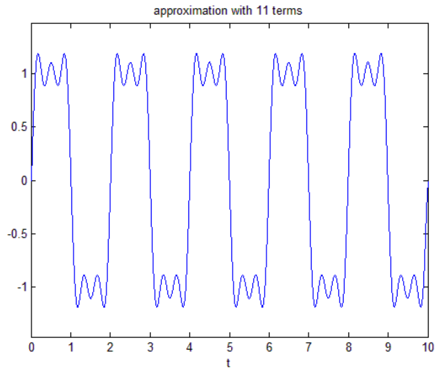
Finally, the complex exponential Fourier series coefficients a_k are calculated and the approximate signal $xx(t)$ is computed and plotted for $0 \leq t \leq 10$, i.e., for time of five periods. As in previous examples, $xx(t)$ is computed and plotted for various number of exponential terms used.

Commands	Results

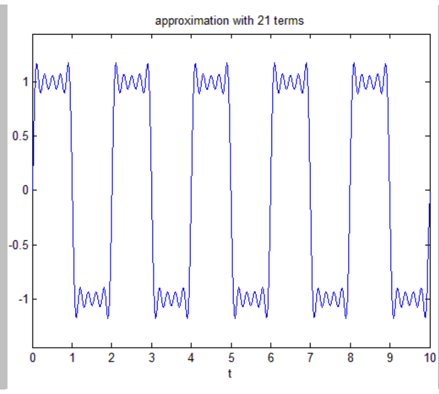
```
k=-2:2;
t0 =0;
T=2;
w=2*pi/T;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T)
xx=sum(a.*exp(j*k*w*t))
ezplot(xx,[0 10])
title('approximation with 5 terms')
```



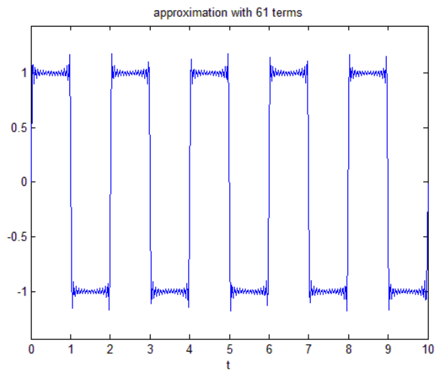

```
k=-5:5;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
xx=sum(a.*exp(j*k*w*t));
ezplot(xx,[0 10])
title('approximation with 11 terms')
```




```
k=-10:10;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
xx=sum(a.*exp(j*k*w*t));
ezplot(xx,[0 10])
title('approximation with 21 terms')
```




```
k=-30:30;
a=(1/T)*int(x*exp(-j*k*w*t),t,t0,t0+T);
xx=sum(a.*exp(j*k*w*t));
ezplot(xx,[0 10])
title('approximation with 61 terms')
```



Tasks

Task 01: The periodic signal $x(t)$ is defined in one period as $x(t) = te^{-t}, 0 \le t \le 6$. Plot in time of four periods the approximate signals using 81 terms of complex exponential form of Fourier series.

Task 02: Plot the coefficients of the complex exponential Fourier series for the periodic signal that in one period is defined by $x(t) = e^{-t^2}, -3 \le t \le 3$.

Task 03: The periodic signal $x(t)$ in a period is given by

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 1 \le t \le 2 \end{cases}$$

Plot in one period the approximate signals using 41 and 201 term of the complex exponential Fourier series. Furthermore, each time plot the complex exponential coefficients.

Task 04: The periodic signal $x(t)$ in a period is given by

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \end{cases}$$

Calculate the approximation percentage when the signal $x(t)$ is approximated by 3, 5, 7, and 17 terms of the complex exponential Fourier series. Furthermore, plot the signal in each case.